Exercise 41

Find the work done by the force field

$$\mathbf{F}(x, y, z) = \langle x - y^2, y - z^2, z - x^2 \rangle$$

on a particle that moves along the line segment from $(0, 0, 1)$ to $(2, 1, 0)$.

Solution

Parameterize the path that the particle moves on by $x = 2t$, $y = t$, $z = 1 - t$ so that

$$\mathbf{r}(t) = \langle 2t, t, 1 - t \rangle, \quad 0 \leq t \leq 1.$$

Calculate the line integral of the force field over the linear path to find the work done.

$$\begin{aligned}
W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^1 \langle (2t) - (t)^2, (t) - (1 - t)^2, (1 - t) - (2t)^2 \rangle \cdot \langle 2, 1, -1 \rangle\, dt \\
&= \int_0^1 \langle 2t - t^2, -t^2 + 3t - 1, -4t^2 - t + 1 \rangle \cdot \langle 2, 1, -1 \rangle\, dt \\
&= \int_0^1 [(2t - t^2)(2) + (-t^2 + 3t - 1)(1) + (-4t^2 - t + 1)(-1)]\, dt \\
&= \int_0^1 (t^2 + 8t - 2)\, dt \\
&= \left(\frac{t^3}{3} + 4t^2 - 2t \right) \Bigg|_0^1 \\
&= \frac{1^3}{3} + 4(1)^2 - 2(1) \\
&= \frac{7}{3}
\end{aligned}$$